Indian Statistical Institute, Bangalore

B. Math. Second Year

First Semester - Optimization

Duration : 3 hours | Date : November 22, 2017

Semester Exam | **Total marks: 50**

Section I: Answer any four and each question carries 6 marks.

1. Prove QR-decomposition for full column rank matrices and prove the decomposition is unique if R is required to have positive entries on the diagonal.

2. If $A = UDV^t$ is the singular value decomposition, prove that $Ax = b$ has a solution if and only if $b \perp U_i$ for all $i > k$ where k is the rank of A.

3. Prove that $\text{Spr}(A)$ has algebraic multiplicity one for a nonnegative irreducible matrix A.

4. Solve by simplex method

$$\begin{array}{rr} \text{Maximize} & 9x_1 + 10x_2 \\ \text{subj} & x_1 + 2x_2 \leq 8 \\ & 5x_1 + 2x_2 \leq 16 \\ & x \geq 0. \end{array}$$

5. Solve the following game: Ruby conceals either a Rs. 1 coin or Rs. 2 coin in her hand; Charm guesses 1 or 2, winning the coin if he guesses the number.

6. Prove the existence and uniqueness of minimum norm least square solution to $Ax = b$.

Section II: Answer any two and each question carries 13 marks.

1. (a) Let s_i be the i-th singular value of A. Prove that $s_i \leq ||A - B||$ for any matrix B with rank $(B) < i$ (*Marks: 7*).

   (b) For $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$, use Perron-Frobenius theory to compute $\lim_{n\to\infty} A^n$.

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2. (a) Let $A \geq 0$ be a primitive matrix, v and u be positive eigenvectors for A and A^t with eigenvalue $\mathrm{Spr}(A)$. If $u^t v = 1$ and $r = \mathrm{Spr}(A)$, prove that $(\frac{A}{r})^n \to vu^t$ exponentially (*Marks: 6*).

   (b) Let A and B be two matrices such that $b_{ij} = a_{ij} + r$. Then show that two strategy vectors p and q are optimal for A if and only if they are optimal for B and value of the game B is value of game A plus r.

3. (a) Describe and justify a method to avoid anticycling in LP (*Marks: 6*).

   (b) State and prove a necessary and sufficient condition in terms of the matrix entries for a 2×2- matrix game to be non-strictly determined.

   (c) Solve the $n \times n$-game $A = I_n$ (*Marks: 3*).